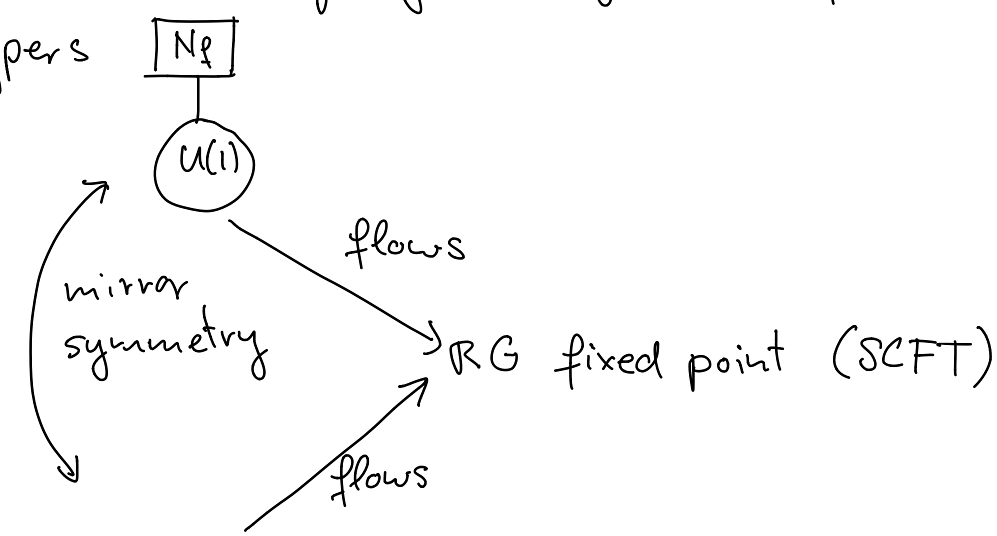
Mirror symmetry for $\mathcal{N}=4$ $U(1)$ gauge theories

3d $\mathcal{N}=4$ theories have $SO(4) = SU(2)_R \times SU(2)_{R'}$ R-symmetry

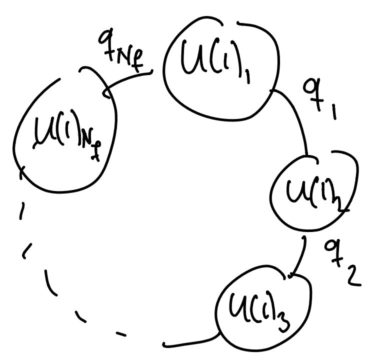
$SU(2)_R$ acts on the Higgs branch

$SU(2)_{R'}$ acts on Coulomb branch

Consider $U(1)$ gauge theory with $N_f > 1$ charged hypers



$\left(U(1)^{N_f}\right)/U(1)$ gauge theory with N_f hypers q_i of charge 1 under $U(1)_i$ and charg -1 under $U(1)_{i+1}$



$\mathcal{N}=4$ vector multiplet: $\mathcal{N}=2$ vector mult. $+$ $\mathcal{N}=2$ chiral mult.

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Giving mass to Ψ $\longrightarrow$ $\mathcal{N}=2$ theory discussed before ($U(1)$ with N_f fl.)
theory A

two ways: $W = m\Psi^2$

or $W = mS\Psi$ $\longleftarrow$ choose this way

$\longrightarrow$ S acts as dynamical FI term

$\xrightarrow{\text{mirror symmetry}}$ $W = S\sum_{i=1}^{N_f} q_i \tilde{q}^i$ in the dual theory

including $N_f - 1$ neutral chiral fields in the $U(1)^{N_f-1}$ vector multiplets and their superpotential couplings:

$$W = \sum_{i=1}^{N_f} S_i q_i \tilde{q}^i \quad \text{theory B}$$

Coulomb branch of A:
parametrized by
$V_\pm$

$\longleftrightarrow$
$V_\pm = N_\pm$

Higgs branch of B:
parametrized by gauge invariant ops
$N_- = q_1 q_2 \cdots q_{N_f}$
$N_+ = \tilde{q}^1 \tilde{q}^2 \cdots \tilde{q}^{N_f}$

Higgs branch of A:
param. by $M^{i}_{\tilde{j}} = Q^i \tilde{Q}_{\tilde{j}}$
with constraint
$M^{i}_{\tilde{j}} M^{l}_{\tilde{k}} = M^{i}_{\tilde{k}} M^{l}_{\tilde{j}}$

$\longleftrightarrow$
$M^{i}_{\tilde{i}} = S_i$
$M^{i-1}_{\tilde{i}} = V_{i,+}$
$M^{i}_{\tilde{i-1}} = V_{i,-}$

Coulomb branch of B:
param. by N_f singlets S_i and $N_f - 1$ dual photons
$V_{i,+} V_{i,-} = S_i S_{i-1}$

Vortex Interpretation

Consider $\mathcal{N}=2$ $U(1)$ with N_f flavors $Q^i, \tilde{Q}_{\bar{i}}$

Have "vortex solution": $Q \sim \sqrt{\zeta}\, e^{\pm i\theta}$, $A_\theta \sim \pm \frac{1}{r}$

BPS bound is satisfied

In $\mathcal{N}=4$ language have $SU(2)_R$ triplet of FI terms:

real ζ_r + complex ζ_c

BPS charge: $Z = \zeta \int r\, d\theta\, A_\theta = \zeta \int d^2x\, \varepsilon^{0\mu\nu} F_{\mu\nu}$

$$= \zeta \int d^2x\, j^0$$

$\to$ on Higgs branch there are massive vortex states of non-zero $\mathcal{J}$-charge

We know that $V_\pm \sim e^{\pm\Phi}$ has $\mathcal{J}$-charge $\pm\zeta$

On Coulomb branch $U(1)_{\mathcal{J}}$ is broken by

$$\langle V_\pm \rangle \neq 0$$

$\to$ "Vortex condensation"

Consider $\mathcal{N}=4$ $U(1)$ with $N_f = 1$

$\zeta \neq 0 \to$ unique vacuum with vortex v_+ and anti-vortex v_- are BPS with $Z = m = \zeta$

in $\mathcal{N}=2$ language:

$Z = \zeta_r$ real vortex mass, $W = \zeta_c v_+ v_-$, $m = \sqrt{\zeta_r^2 + |\zeta_c|^2}$ (non-BPS)

$$\Big\downarrow\; W = S\psi$$

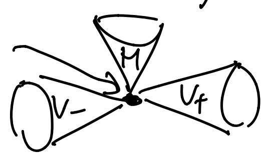
$\mathcal{N}=2$ with $\xi_c = S$

eq.o.m. for ψ gives: $S = -Q\tilde{Q} = -M$

(to get the form in last lecture, substitute

$v_\pm = V_\pm \longrightarrow W = M V_+ V_-$)

Vortices are BPS at $M=0$



Consider now $N_f = 2$, $\mathcal{N}=2$:

$M^i{}_{\tilde{j}}$ form a $(2,2)$ of $SU(2)\times SU(2)$

vortices: $V_\pm$

mirror has photon and corresponding dual scalar $\gamma \longrightarrow U(1)_\gamma \longleftrightarrow T_3 \in SU(2)_{diag} \subset SU(2)\times SU(2)$

T_3 rotates $M^2{}_1$ and $M^1{}_2$ in opposite directions, leaves $M^i{}_i$ unchanged

$$\longrightarrow \quad \begin{array}{ccc} M^2{}_1 & & M^1{}_2 \\ \updownarrow & \text{mirror} & \updownarrow \\ W_+ \sim e^{\Phi + i\gamma} & & W_- \sim e^{-\Phi - i\gamma} \end{array}$$

$\rightarrow$ $SU(2)\times SU(2)$ flavor symmetry appears in the mirror throug $M^1{}_1 = S_1$ and $M^2{}_2 = S_2$ which combine with $W_\pm$ to form $(2,2)$